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Real space renormalisation group study of bond–site percolation on an anisotropic triangular lattice

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Abstract. We consider the problem of bond–site percolation on a triangular lattice in which bonds are separated into two classes with different occupation probabilities. Pure site and pure bond percolation on the triangular and square lattices are special cases of this system. We use an approximate real space renormalisation group treatment to identify the critical surface and we present evidence for the universality of critical exponents for site, bond and mixed bond–site percolation, except at a single point in the space which corresponds to percolation in one dimension.

1. Introduction

The extension of the usual percolation models to allow both bonds and sites to be occupied at random and independently (bond–site percolation) seems to have been first discussed by Frisch and Hammersley (1963). Bond–site percolation has been considered (Hammersley and Welsh 1980) as a model of the spread of disease in a biological population (where the site and bond densities correspond respectively to the susceptibility and infectability of individuals in the population), and as a model of polymer gelation in the presence of solvent molecules which do not contribute to the polymerisation process (Stauffer 1981). Relatively little work has appeared in this area. Bond–site percolation on the square lattice has been studied both by series analysis methods (Agrawal *et al* 1979) and by real space renormalisation group techniques (Nakanishi and Reynolds 1979). In addition, Napiorkowski and Hemmer (1980) have considered bond–site percolation on a square lattice with nearest-neighbour and next-nearest-neighbour bonds, in which the two classes of bonds can have different occupation probabilities. Hammersley and Welsh (1980) have discussed some inequalities between the percolation probability in a bond–site percolation process and the percolation probabilities for the corresponding pure bond and pure site processes.

In this paper we consider bond–site percolation on a triangular lattice in which bonds are divided into two classes with different occupation probabilities. The triangular lattice can be deformed into a square lattice with an additional diagonal bond in each square (see figure 1). If we allow sites to be occupied independently with probability s, 'horizontal' and 'vertical' bonds to be occupied independently with probability b, and 'diagonal' bonds to be occupied with probability d then, by varying s, b and d, we can move in this three-parameter space from one well known percolation

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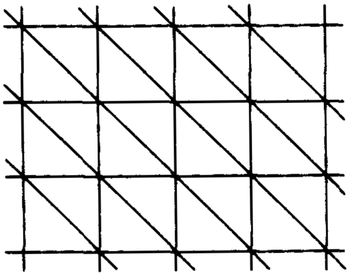

Figure 1. The triangular lattice deformed to give a square lattice with additional diagonal bonds.

system to another. Thus $b = d = 1$ corresponds to site percolation (with site density s) on the triangular lattice, $b = 1$, $d = 0$ to site percolation on the square lattice, $s = 1$, $d = 0$ to bond percolation on the square lattice and $s = 1$, $b = d$ to bond percolation on the triangular lattice. In addition, setting $b = 0$ uncouples the diagonal bonds into a series of linear chains and leads to percolation on a line. The special case of $b = 1$ has been studied using Monte Carlo methods by Hoshen *et al* (1979) and by series analysis methods in the following paper (Torrie *et al* 1982).

An investigation of this bond–site percolation process will give information on the universality of critical exponents for the various special cases discussed above and for the more general process. In this paper we construct a real space renormalisation group scheme and present evidence that the exponents are universal except at the special point corresponding to percolation in one dimension.

2. Renormalisation group scheme

We first construct a 'cell' which, by translation in two perpendicular directions, generates the lattice. We could choose any such 'cell', but the cell shown in figure 2

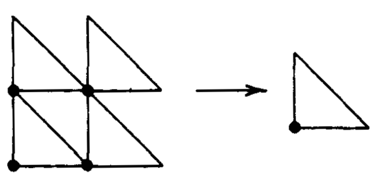

Figure 2. Renormalisation scheme for a single cell.

is small enough to be easily analysed and yet large enough to retain the essential features of the problem. This renormalises into a site, two 'b' bonds and one 'd' bond. The site renormalisation (shown in figure 3) is straightforward and the renormalised site density (s') is the probability that at least one path exists crossing the cell, i.e.,

$$s' = s^2d + 4s^2b - 4s^3bd - 4s^3b^2 + 2s^3b^2d - 2s^4b^2 + 4s^4b^2d + 4s^4b^3 - 4s^4b^3d - s^4b^4 + s^4b^4d. \tag{2.1}$$

The first two terms represent the probabilities of the five single bond paths and the remainder of the expression follows by application of the inclusion–exclusion principle.

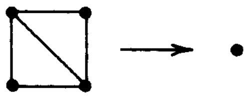

Figure 3. Site renormalisation scheme.

When we consider bond renormalisation there is an additional complication since some paths must renormalise to 'b' bonds and the remainder to 'd' bonds. This division is somewhat arbitrary but we have chosen the following system. Any path from a source site (designated as a cross) in figure 4 to a site at the opposite side of the cell (open circles) renormalises into a 'b' bond terminating in a site, thus

$$\begin{aligned} s'b' = {} & 2s^2bd + 2s^2b^2 - s^2b^2d + s^3bd^2 + s^3b^2d - 4s^3b^2d^2 \\ & + 2s^3b^3 - 7s^3b^3d + 5s^3b^3d^2 - 2s^3b^4 + 4s^3b^4d \\ & - 2s^3b^4d^2 - s^4b^2d^2 - 3s^4b^3d + 4s^4b^3d^2 - 3s^4b^4 \\ & + 8s^4b^4d - 5s^4b^4d^2 + 2s^4b^5 - 4s^4b^5d + 2s^4b^5d^2. \end{aligned} \tag{2.2}$$

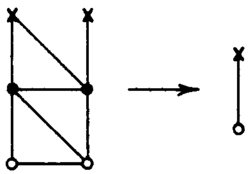

Figure 4. Renormalisation of vertical bonds.

Similarly, any path crossing the cell in figure 5 from the source site (or cross) to the sink site (or open circle) renormalises to a 'd' bond terminating in a site. Hence

$$s'd' = \{2s^2b^2 + sd - 2s^2b^2d - s^3b^4 + s^3b^4d\}^2. \tag{2.3}$$

Equations (2.1)–(2.3) define the renormalisation group transformation in the (s, b, d) space and the fixed points of this transformation are (0, 0, 0), (1, 1, 1), (1, 0, 1)

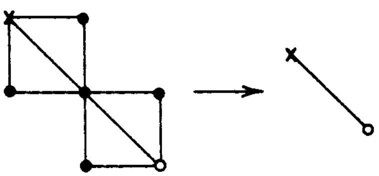

Figure 5. Renormalisation of diagonal bonds.

and (0.7482, 0.5709, 0.2384). When we examine the flow patterns we find that the first two fixed points are trivial (attractive) fixed points. The third point turns out to be an isolated fixed point and corresponds to one-dimensional percolation, since setting $b = 0$ decouples the 'd' bonds into a set of independent linear chains. Percolation only occurs in this one-dimensional system for $s = d = 1$. The fourth fixed point (F) lies on the critical surface which divides the gel region from the sol region. Flows from points in the sol region are into (0, 0, 0) and from points in the gel region are into (1, 1, 1). All points on the critical surface (except (1, 0, 1)) are attracted to F, indicating that the critical behaviour will be universal, except at (1, 0, 1).

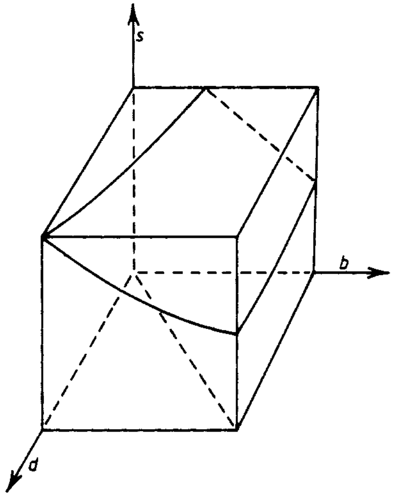

Figure 6. The critical surface separating the sol region from the gel region in the (s, b, d) unit cube.

The critical surface is shown in figure 6. Like the relevant fixed points, it is obtained numerically, by following flows from points on the surface of the unit (s, b, d) cube. The points of intersection of the critical surface with appropriate edges of the cube give estimates of the critical percolation densities for the square bond, square site and triangular site problems. These are respectively 0.45, 0.52 and 0.47. These are somewhat lower than the exact values of $\frac{1}{2}$ for the square bond (Sykes and Essam 1964, Kesten 1980) and triangular site problems (Sykes and Essam 1964) and the series estimate of 0.593 (Sykes *et al* 1976) for the square site problem. Similarly, the intersection of the critical surface with the line $s = 1$, $b = d$ is an estimate of the critical density of the triangular bond problem. This value of 0.34 is lower than the exact value (Sykes and Essam 1964) of $2\sin(\pi/18) \simeq 0.347$. On the whole, the level of agreement with exact results is very satisfactory.

After linearising the transformation about F, we find that the eigenvalues of the (linearised) transformation are $\lambda_1 = 1.721$, $\lambda_2 = 0.609$ and $\lambda_3 = 0.246$. Since the scale factor for the transformation is 2, the pair connectedness length exponent (ν) is given by

$$\nu = \log 2/\log \lambda_1 = 1.28. \qquad (2.4)$$

This value should be compared with recent renormalisation group estimates for two-dimensional bond–site percolation of 1.47 (Nakanishi and Reynolds 1979) and 1.22 (Napiorkowski and Hemmer 1980) and the conjecture of den Nijs (1979) of $\frac{4}{3}$. The relevant eigenvalues, in principle, give correction terms to the leading exponent ν, but given the approximate nature of the model we don't feel that much weight should be attached to the numerical value of correction exponents.

Turning now to the fixed point at (1, 0, 1), corresponding to one-dimensional percolation, we find that the eigenvalues are 1, 2 and 3. The smallest relevant eigenvalue gives $\nu = \log 2/\log 2 = 1$ in agreement with exact results. The marginal eigenvalue is not attractive, as off the critical surface all flows go to (0, 0, 0) or (1, 1, 1), depending upon whether they start in the sol or gel region, while, on the critical surface, all flows are attracted to the physical fixed point (F). Thus the point (1, 0, 1) is an isolated, repulsive fixed point, in a distinct universality class. This, of course, is precisely what one would expect from physical considerations.

3. Discussion

We have presented a real space renormalisation group treatment of bond–site percolation on an anisotropic triangular lattice. The transformation has two non-trivial fixed points, one of which is an isolated fixed point corresponding to percolation in one dimension, and the other governs the critical behaviour of the remainder of the three-parameter space. This is strong evidence for universality and, in particular, suggests that bond and site percolation on the square and triangular lattices will have the same critical exponents, as will mixed bond–site percolation on these lattices.

Our estimates of the critical densities for bond and site percolation on the square and triangular lattices, although all somewhat low, are in reasonable agreement with exact values or with values derived by series analysis. They could perhaps be improved by eliminating certain paths in our renormalisation scheme, but there seems no compelling physical argument to warrant such elimination. Our estimate of the pair connectedness exponent (ν) is reasonably close to the conjectured value of den Nijs.

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